Investigating the Effect of Small Sample Process Capability Index Under Different Bootstrap Methods

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***Abstract*—In the quality control of multi-variety and small-batch products, the calculation of the process capability index is particularly important. However, when the sample size is not enough, the process distribution cannot be judged, if the traditional method is still used to calculate the process capability index; there will be misapplication or misuse. In this paper, the Bootstrap method is introduced into the estimation of process capability index and the calculation of its confidence interval by using Standard Bootstrap (SB), Percentile Bootstrap (PB), Percentile-t Bootstrap (PTB) and Biased-corrected Percentile Bootstrap (BCPB) methods were used to analyze and compare the process capability index. It is found that in symmetric distribution, only the sample size has a significant effect on the length of the confidence interval； but in asymmetric distribution, sample size and Bootstrap methods are both significant factors affecting the length of confidence interval.**

Keywords—Process capability indices; bootstrap; confidence interval; small samples

I. Introduction

Process capability analysis is an important part of statistical process control activities for continuous improvement. Process Capability Index (PCI) is designed to provide a general language for quantifying its performance; it is a dimensionless function of the process parameters and specifications. PCI is widely used in traditional manufacturing industries, but with the production mode has changed from single variety and large batch to multi-variety and small batch, and the number of parts of the same specification produced under the same process and similar production conditions has become less and less, which brings difficulties to the process capability analysis and the calculation of indicators and statistical inference. When the sample size is insufficient, the central limit theorem cannot be used to calculate the process capability index because it is impossible to judge the distribution of process data. If the traditional method is still used for calculation, it is easy to misunderstand and misuse.

Process Capability Indices (PCIs) are considered as a practical tool by many advocates of statistical process control in industry. They are used to determine whether a manufacturing process is capable of producing with dimensions within a specified tolerance range. The process indices *Cp* and *Cpk* [1] are used for unit-less measures that relate the natural process tolerance (6σ), upper and lower specification limits. Chan et al. [2] developed *Cpm* that incorporates a target value for the process. Taguchi [3] and Chou et al. [4] provided tables for constructing 95 percent lower confidence limits for both *Cp* and *Cpk*. Their tables for limits on *Cpk*, however, are conservative and an approximation presented by Bissel [5] is recommended instead. Boyles [6] provided an approximate method for finding lower confidence limits for *Cpm*. The calculation of all these lower confidence limits assume a normally distributed process and many real world processes are not normally distributed and this departure from normality may be hard to detect. This could potentially affect both the estimates of the indices and the confidence limits based on these estimates. Efron [7] introduced and developed the non-parametric, but computer intensive estimation method called Bootstrap. Bootstrap method [8] replaces theoretical analysis with computer simulation technology, and replaces real distribution with statistical empirical distribution. It is effective for statistical analysis and process capability analysis under small sample conditions. Therefore, Bootstrap method can be introduced into point estimation and confidence interval calculation of *Cp* and *Cpk* under small sample conditions.

The rest of this paper is consisted of as: Section II presents the related works. Section III and IV realizes the estimation of *Cp* and *Cpk* based on Bootstrap method, and then experimental results are discussed in Section IV. Finally, this paper concludes in Section VI. Our study shows that in symmetric distribution, only the sample size has a significant effect on the length of the confidence interval, Bootstrap methods has no significant effect on the length of confidence interval. But in the skewed distribution, sample size and Bootstrap methods are both significant factors affecting the length of confidence interval.

II. Related Work

Bootstrap method is very popular in modern statistics. Especially after the rise of big data, the effect of estimating the mean or variance of statistics with small samples is ideal. Its simulation result is very close to the real result, and it is often used to solve some situations that cannot be broken through in theory. The core idea of Bootstrap method is to replace theoretical analysis with computer simulation technology, that is, to extract the same number of samples from the original samples by repeated sampling technology, and replace the real distribution with its statistics. Bootstrap method can be used for the hypothesis testing and interval estimation problems of location parameter with unknown scale parameter and skewness parameter, it provides the satisfactory performances under the senses of Type I error probability and power in most cases regardless of the moment estimator or ML estimator [9]. By repeating the above process and calculating its mean or variance, the empirical distribution of statistics is substituted for the real distribution.

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Owing to these benefits, application of Bootstrap method in the process capability index includes:Bootstrap confidence intervals for indices such as *Cp* [10-13], *Cpk* [14, 15], *Cpm* [15], and *Cpmk* [16] are established for PCIs. Its applications can also be divided into: Using Bootstrap sampling to estimate the multivariate [17] or multiple process streams [13] process capability indices; using different methods of estimation to construct Bootstrap confidence intervals of generalized process capability index *Cpyk* [18]. Especially when data is non-normal, the Bootstrap confidence intervals of different distribution types of non-normal data were studied: such as the Modified Process Capability Index for Wei-bull distribution [19], Parametric and non-parametric bootstrap confidence intervals of C_{Npk} for exponential [20] and exponential power distribution [21], and so on. In addition, its application can also be seen in the case of small samples [22, 23]. Bootstrap method replaces theoretical analysis with computer simulation technology, and replaces real distribution with statistical empirical distribution. It is effective for statistical analysis and process capability analysis under small sample conditions. Therefore, Bootstrap method can be introduced into point estimation and confidence interval calculation of *Cp* and *Cpk* under small samples.

III. Estimation of C_P and C_{PK}

A. Definition and Estimation of PCIs

The capability of a process is frequently measured by a process capability index (PCI) which is designed to provide a common and easily understood language for quantifying its performance, and is a dimensionless function of process parameters and specifications. Let *USL* and *LSL* be the upper and lower specification limits, respectively. If the process follows or approximately follows a normal distribution, the statistical characteristics of the traditional process capability index can be calculated, including point estimation, confidence interval, and estimated distribution characteristics.

If the process follows or approximately follows a normal distribution, using the above two sample statistics, point estimates of PCIs such as *Cp & Cpk* [1] , and *Cpm* [2] can be calculated.

The definition of *Cp* is:

$$C_p = \frac{USL - LSL}{6\sigma} \tag{1}$$

The estimation of *Cp* is:

$$\hat{C}_p = \frac{USL - LSL}{6S} = \frac{\sigma}{S} C_p \tag{2}$$

Wherein, *USL* and *LSL* are the upper and lower specification limits of the process.

Since the index *Cp* does not take into account the location of the process mean (μ), the index *Cpk is defined*:

$$\hat{C}_{pk} = min\{ \frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma} \} \tag{3}$$

If the sample size is large and the data follows a normal distribution, a point estimate of the process capability indicator can be obtained by calculating the mean and standard deviation of the sample. They can be represented by the following statistics:

μ is represented by the sample mean $\overline{X}$:

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{4}$$

σ^2 is represented by the sample variance S^2 :

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 \tag{5}$$

Wherein, X_i is the i-th observation value, $\overline{X}$ is the sample mean, and *n* is the sample size. $\overline{X}$ and S^2 are unbiased estimators of population mean μ and population variance σ^2.

The expected value for obtaining the estimated value of the exponent *Cp* by calculating its *r-th* moment $\hat{C}_p$ is:

$$E[\hat{C}_p^r] = [\frac{n-1}{2}]^{\frac{r}{2}} \frac{\Gamma[\frac{n-1-r}{2}]}{\Gamma[\frac{n-1}{2}]} C_p^r \tag{6}$$

So when *r*=1, the mean of the estimated value of the index *Cp* [24] is:

$$E(\hat{C}_p) = \frac{1}{b_f} C_p \tag{7}$$

And b_f is the correction coefficient, $b_f = \sqrt{\frac{n-1}{2}} \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}$.

So when *r*=2, the variance of the estimated value of the index *Cp* [24] is :

$$Var(\hat{C}_p) = (\frac{n-1}{n-3} - \frac{1}{b_f^2}) C_p^2 \tag{8}$$

Therefore, based on (6), it can be obtained that if the process follows or approximately follows a normal distribution, then for *n* samples in the population, the statistic $\frac{(n-1)s^2}{\sigma^2}$ follows χ^2 distribution with *n-1* degrees of freedom, denoted as: $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$.

So:

$$C_p = \hat{C}_p \sqrt{\frac{\chi^2_{n-1}}{n-1}} \tag{9}$$

When the significance level is α , the $100(1-\alpha)\%$ confidence interval for *Cp* [25] is:

$$[\hat{C}_P\sqrt{\frac{\chi^2_{(\alpha/2,n-1)}}{n-1}},\ \hat{C}_P\sqrt{\frac{\chi^2_{(1-\alpha/2,n-1)}}{n-1}}] \qquad (10)$$

The same is true of the $100(1-\alpha)\%$ confidence interval for *Cpk* [25]:

$$[\hat{C}_{pk}(1+z_{\alpha/2})\sqrt{\frac{1}{9n\hat{C}^2_{pk}}+\frac{1}{2(n-1)}},\hat{C}_{pk}(1+z_{1-\alpha/2})\sqrt{\frac{1}{9n\hat{C}^2_{pk}}+\frac{1}{2(n-1)}}] \qquad (11)$$

B. Analysis of the Influence of Sample Size

It can be seen from Eq. (6) to Eq. (11) that sample size *n* has an impact on the statistical estimator of the process capability index.

1) Impact of sample size on the correction coefficient b_f *:* As shown in Table I: With the increase of sample size n, the closer the value of $1/b_f$ is to 1, the closer the point estimate is to the true value of Cp. When n=400, b_f value loses significance, and the corresponding PCI estimation is also meaningless. It follows that the appropriate sample size is usually 100 or 200. There is no need to adopt full sampling method or blindly increase sample size.

2) Impact of sample size on the confidence interval of Cp: The confidence interval of Cp is shown in Table II, when it follows or approximately follows a normal distribution at a given value. As shown in the Table II, with a given value $\hat{C}_p$, the width of the confidence interval becomes shorter as the sample size n increases. When the sample size n is specified, the larger the $\hat{C}_p$ value is, the longer the confidence interval length is. Therefore, when the data follows or approximately follows a normal distribution, in order to obtain a tighter confidence interval for Cp, it is necessary to consider both the $\hat{C}_p$ value and the sample size n in order to obtain a more stringent Cp confidence interval.

Wherein，*Lc* and *Uc* are the lower and upper confidence limits.

3) Impact of sample size on the confidence interval length of Cpk: As shown in Table III: With reference to the conclusions of the above Table I and Table II, considering the economy of sampling and the accuracy of parameter confidence intervals, we only analyzed the influence of sample size within 200 on the confidence intervals and interval widths of *Cpk*.

Similar to the point estimation and interval estimation of *Cp*, when the data obey or approximate obey the normal distribution, the confidence interval and interval width of *Cpk* are affected by both the sample size *n* and the *Cpk* estimator. When $\hat{C}_{pk}$ is specified, the larger the sample size is, the more accurate the interval estimation and the shorter the width of the confidence interval are. When the sample size is specified, the larger the $\hat{C}_{pk}$ value is, the longer the corresponding confidence interval width is. Here *CIL* indicates the confidence interval length.

Based on the analysis shown in Table I to Table III, we can find that the traditional process capability analysis is based on the process obeying or approximately obeying the normal distribution. In order to ensure the reliability of parameter estimation, a sufficient sample size is usually required, i.e., *n*=100 or 200. This means that in traditional analysis we need a large sample. In other words, when the sample size is small, it is impossible to accurately judge the type of distribution of the data. At this point, if the traditional parameter estimation method is still used to analyze the process capability, the calculated *PCI* is not accurate and the *CIL* is longer. We'll get the wrong conclusions. To solve these problems, we can use the Bootstrap method to calculate the confidence interval of the process capability indicator.

TABLE I. IMPACTS OF SAMPLE SIZE ON B_F

n	5	6	7	8	9	10	20	30	40
b_f	1.253314	1.189416	1.151243	1.125869	1.107784	1.094242	1.041764	1.026826	1.019759
n	50	60	70	80	90	100	200	300	400
b_f	1.015639	1.01294	1.011036	1.009621	1.008527	1.007656	1.003789	1.002517	---

TABLE II. CONFIDENCE INTERVALS OF $\hat{C}_p$

n [Lc,Uc]	$\hat{C}_p$ =1	$\hat{C}_p$ =1.33	$\hat{C}_p$ =1.5	$\hat{C}_p$ =1.67	$\hat{C}_p$ =2.0
5	[0.3480, 1.6691]	[0.4628, 2.2199]	[0.5220, 2.5037]	[0.5812, 2.7874]	[0.6960, 3.3382]
10	[0.5478, 1.4538]	[0.7286, 1.9336]	[0.8217, 2.1807]	[0.9148, 2.4279]	[1.0956, 2.9076]
50	[0.8025, 1.1971]	[1.0673, 1.5921]	[1.2037, 1.7957]	[1.3402, 1.9992]	[1.0050, 2.3942]
100	[0.8608, 1.1389]	[1.1449, 1.5147]	[1.2912, 1.7084]	[1.4375, 1.9020]	[1.7216, 2.2778
200	[0.9018, 1.0981]	[1.1994, 1.4605]	[1.3527, 1.6472]	[1.5060, 1.8338]	[1.8036, 2.1962]
500	[0.9380, 1.0620]	[1.2235, 1.4365]	[1.4070, 1.5930]	[1.5665, 1.7735]	[1.8760, 2.1240]
1000	[0.9561, 1.0438]	[1.2716, 1.3883]	[1.4341, 1.5657]	[1.5967, 1.7432]	[1.9122, 2.0876]

TABLE III. IMPACTS OF SAMPLE SIZE ON CONFIDENCE INTERVALS LENGTH OF C_{PK}

n [Lc,Uc]	$\hat{C}_{pk}=1$	CIL	$\hat{C}_{pk}=1.5$	CIL	$\hat{C}_{pk}=1.67$	CIL
5	[0.2480, 1.7520]	1.5041	[0.4203, 2.5797]	2.1595	[0.6231 ,2.8577]	2.2346
10	[0.4939, 1.5061]	1.0121	[0.7769, 2.2231]	1.4462	[0.9308 2.3932]	1.4623
50	[0.7815, 1.2185]	0.4370	[1.1890, 1.8110]	0.6221	[1.3560, 2.0195]	0.6635
100	[0.8461, 1.1539]	0.3077	[1.2811, 1.7189]	0.4378	[1.4502, 1.8985]	0.4483
200	[0.8914, 1.1086]	0.2171	[1.3456, 1.6544]	0.3089	[1.4976, 1.8145]	0.3168

IV. ESTIMATION OF C_P AND C_{PK} BASED ON BOOTSTRAP METHOD

A. Introduction of Bootstrap

The Bootstrap method is to repeatedly resample the original sample, extract *B* replacement samples with random (RSWR) with sample size *n* from sample S_0, and express them with S_i^* (subscript *i* represents the *i-th* time resampling). Where, $S_i^*=\{x_1^*,...,x_n^*\}$ represents a simple RSWR extracted from S_0, S_i^* is called Bootstrap sample. For each subsample S_i^*, its *T* statistic is calculated and expressed by $\{t_1^*,t_2^*,\ldots t_B^*\}$ respectively.

Assuming there is a random sampling sequence $S_0=\{x_1,...,x_n\}$ with a length of *n* from a completely uncertain distribution. Where x_i is the independent random sampling of the distribution, where t_i represents the value of a specific sample statistic *T* .

The distribution of the statistic *T* is called the Empirical Bootstrap Distribution (EBD), where *B* is the sample size. When *B* is large enough, an approximation of the statistic *T* can be obtained by repeated sampling from S_0. In this way, the Bootstrap method can be used for statistical simulation of small samples, so as to obtain statistical estimation of unknown distribution and unknown parameters. Generally, we assumed *B* = 1000 bootstrap re-samples.

Bootstrap methods include Standard Bootstrap (SB), Percentile Bootstrap (PB), Biased corrected Percentile Bootstrap (BCPB), Percentile *T* Bootstrap (PTB), and Biased corrected and accelerated Bootstrap (BCa). Scholars [10-19, 21] mostly used the two of four methods and compared the Bootstrap confidence interval of PCIs. It is difficult to implement the BCa method [20], so its application is less. Therefore, this paper also uses the first four Bootstrap methods.

B. Bootstrap Confidence Intervals with Four Methods

Based on the original random samples $x_1, x_2,..., x_n$ with sample size *n*, construct *B* new Bootstrap samples $x_1^*, x_2^*,..., x_n^*$, i.e. $S_i^*=\{x_1^*,...,x_n^*\}$. Calculate the *Cp*values for each sub sample S_i^*, denoted by $\{C_1^*,C_2^*,\ldots,C_B^*\}$ respectively. By arranging the values in ascending order, the empirical probability distribution of *B T*-values can be obtained, which is called the Bootstrap Empirical Distribution (EBD). Taking *B*=1000 and using the Bootstrap method to repeat sampling from small sample S_0, statistical estimates of *Cp* can be obtained.

Here $\hat{C}_p$ and $\hat{C}_{pk}$ represents the estimate of *Cp* and *Cpk*, $\hat{C}_p^*(i)$ and $\hat{C}_{pk}^*(i)$ represent the sequential estimator of the process capability index calculated from 1000 Bootstrap random replacement samples. The sample mean calculated from these 1000 Bootstrap estimators are:

$$\overline{C}_p^* = \frac{1}{1000}\sum_{i=1}^{1000}\hat{C}_p^*(i) \tag{12}$$

$$\overline{C}_{pk}^* = \frac{1}{1000}\sum_{i=1}^{1000}\hat{C}_{pk}^*(i) \tag{13}$$

The standard deviation of the samples are:

$$S_{cp}^* = \sqrt{\frac{1}{999}\sum_{i=1}^{1000}\left[\hat{C}_p^*(i)-\overline{C}_p^*\right]^2} \tag{14}$$

$$S_{cpk}^* = \sqrt{\frac{1}{999}\sum_{i=1}^{1000}\left[\hat{C}_{pk}^*(i)-\overline{C}_{pk}^*\right]^2} \tag{15}$$

When the distribution of $\hat{C}_p$ and $\hat{C}_{pk}$ follows or approximates the normal distribution, the statistics calculated in Eq. (14) and Eq. (15) are essentially estimators of the standard deviations of *Cp* and *Cpk* .

1) Confidence interval based on SB[8,14]: When the significance level is *α*, the standard Bootstrap confidence intervals for $100(1-\alpha)\%$ of the process capability index *Cp* and *Cpk* are respectively:

$$\left[\hat{C}_p - Z_{1-\frac{\alpha}{2}}S_{cp}^*,\ \hat{C}_p + Z_{1-\frac{\alpha}{2}}S_{cp}^*\right] \tag{16}$$

$$\left[\hat{C}_{pk} - Z_{1-\frac{\alpha}{2}}S_{cpk}^*,\ \hat{C}_{pk} + Z_{1-\frac{\alpha}{2}}S_{cpk}^*\right] \tag{17}$$

Wherein, $Z_{1-\frac{\alpha}{2}}$ is the *1-α/2* percentile of the standard normal distribution.

2) Confidence interval based on PB [8,14]

$$\left[\hat{C}_p^*(\frac{\alpha}{2}B),\hat{C}_p^*(1-\frac{\alpha}{2})B)\right] \quad (18)$$

$$\left[\hat{C}_{pk}^*(\frac{\alpha}{2}B),\hat{C}_{pk}^*(1-\frac{\alpha}{2})B)\right] \quad (19)$$

Wherein, α/2 and (1-α/2) are the upper and lower bounds of the confidence intervals for the statistics *Cp* and *Cpk* at the (1−α) confidence level, respectively, i.e.

3) Confidence interval based on BCPB[8,15]

$$\left|\hat{C}_p^*(P_L B),\hat{C}_p^*(P_U B)\right| \quad (20)$$

$$\left|\hat{C}_{pk}^*(P_L' B),\hat{C}_{pk}^*(\dot{P}_U' B)\right| \quad (21)$$

Wherein, $P_0 = P_r(\hat{C}_p^* \le \hat{C}_p)$, $P_0' = P_r(\hat{C}_{pk}^* \le \hat{C}_{pk})$, and $Z_0 = \Phi^{-1}(P_0)$, $Z_0' = \Phi^{-1}(P_0')$, $P_L = \Phi(2Z_0 - Z_{1-\frac{\alpha}{2}})$, $P_L' = \Phi(2Z_0' - Z_{1-\frac{\alpha}{2}})$, $P_U = \Phi(2Z_0 + Z_{1-\frac{\alpha}{2}})$, and $P_U' = \Phi(2Z_0' + Z_{1-\frac{\alpha}{2}})$.

4) Confidence interval based on PTB [8,14]

$$\left[\hat{C}_{p,\ pk} - t^*(\frac{\alpha}{2}B)\times S_{p,\ pk}^*,\hat{C}_{p,\ pk} + t^*(\frac{\alpha}{2}B)\times S_{p,\ pk}^*\right] \quad (22)$$

$$\left[\hat{C}_{pk} - t^*(\frac{\alpha}{2}B)\times S_{cpk}^*,\hat{C}_{pk} + t^*(\frac{\alpha}{2}B)\times S_{cpk}^*\right] \quad (23)$$

Wherein，*Scp*∗is the sample standard deviation of { $\hat{C}_{py}^{*(i)}$, i = 1, 2, …, B } , that is $S^* = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{C}_{p,\ pk}^{*(i)} - \bar{\hat{C}}_{p,\ pk}^*)^2}$, where $\bar{\hat{C}}_{p,\ pk}^* = \frac{1}{B}\sum_{i=1}^{B}\hat{C}_{p,\ pk}^{*(i)}$.

In addition, $\hat{t}^{*(\tau)}$ is the τ percentile of $\left\{\frac{\hat{C}_{p,\ pk}^{*(i)} - \hat{C}_{p,pk}}{S^*}\right\}$; i=1, 2, ..., B. i.e., $\hat{t}^{*(\tau)}$ is such that $\frac{1}{B}\sum_{i=1}^{B} I(\frac{\hat{C}_{p,\ pk}^{*(i)} - \bar{\hat{C}}_{p,\ pk}}{S^*} \le \hat{t}^{*(\tau)}) = \tau$, 0<τ<1.

C. Calculation of CIL of Cp and Cpk under Bootstrap Methods

Based on the above principles, enough samples are obtained by repeated sampling from small samples, assuming that they obey different distributions; date should be transformed to normal firstly before calculating PCI.

For comparison, the original distribution is transformed so that the different types of distributions have the same mean and variance. To be more general, the data generated by the simulation is processed centrally, that is, standard normal processing. Assuming that different data distributions have the same mean and variance, the data can be transformed accordingly:

$$\frac{Y - \mu^*}{\sigma^*} = \frac{X - \mu_0}{\sigma_0} \quad (24)$$

where, μ^* , σ^* , μ_0 and σ_0 are the mean and standard deviation of the expected output distribution and the original distribution, respectively. Table IV summarizes the means and variances of some common distributions. These parameters will be used in the data normal transformation.

To be more general, we choose several typical distributions, including symmetric distributions such as the normal distribution and the heavy-tailed distribution t_5, and asymmetric distributions such as the moderately right-skewed distribution χ_5^2 and the slightly skewed distribution *logn*(0, 0.4).

TABLE IV. MEAN AND VARIANCE OF DIFFERENT DISTRIBUTION

Distributions	Mean	Variance	Statistics
Normal distribution	μ	σ^2	μ：Mean σ^2：variance
Exponential distribution	1/λ	1/λ^2	λ：Threshold
χ2 distribution	n	2n	n：Freedom
T distribution	0	n/(n-2)	n：Freedom
F distribution	v/(v-2)	$\frac{2v^2(u+v-2)}{u(v-2)^2(v-4)}$, $v>4$	u&v：first & second degree of freedom
Log-normal distribution	e$^{(\mu+\sigma\hat{}2)/2}$	e$^{2\mu+\sigma\hat{}2(e\sigma\hat{}2-1)}$	μ：Mean σ^2：variance

Firstly, to obtain a different distribution type with a mean of 0 and a standard deviation of 1, you can transform the data based on Eq. (24) as follows:

$$X_1 \sim t_5 \Rightarrow Y_1 = \sqrt{\frac{3}{5}} X_1 \tag{25}$$

$$X_2 \sim \chi_5^2 \Rightarrow Y_2 = \sqrt{\frac{1}{10}} X_2 + \frac{5}{\sqrt{10}} \tag{26}$$

$$X_3 \sim Logn(0{,}0.4) \Rightarrow Y_3 = \frac{1}{e^{0.08}\sqrt{e^{0.16}-1}} X_3 - \frac{1}{\sqrt{e^{0.16}-1}} \tag{27}$$

Secondly, based on the new distribution generated by the above three transformations, using four Bootstrap methods, the confidence interval and *CIL* value at α=0.1 can be obtained through simulation. The simulation results and analysis are shown in Section IV.

V. Experimental Analysis

Based on the calculation results in Section IV, the influence of sample size *n* and different Bootstrap methods on the length of confidence interval is analyzed.

A. When the Data Follows Symmetrical Distribution

In order to analyze the factors affecting CIL, we first carried out ANOVA and drew images to compare the difference of CIL under different Bootstrap methods.

Through ANOVA, we found that only the sample size n had a significant effect on *CIL* (its *P* value less than 0.05), while the Bootstrap method had no significant effect on *CIL* (its *P* value greater than 0.05). In addition, combined with comparison graph analysis in Fig. 1, regardless of which Bootstrap method is used, *CIL* decreases as the sample size *n* increases.

TABLE V. Simulation Result Under Distribution with Four Bootstrap Method

Distribution		Symmetrical Distribution				Asymmetrical Distribution			
		Normal distribution		t_5		χ^2		***logn*(0, 0.4)**	
Method	n	[Lc ,Uc]	CIL	[Lc, Uc]	CIL	[Lc, Uc]	CIL	[Lc, Uc]	CIL
SB	10	[0.4489, 1.5213]	1.0724	[0.3602, 1.7573]	1.3971	[0.2552, 1.7442]	1.789	[0.3655, 1.7383]	1.3728
	20	[0.5335, 1.3548]	0.8213	[0.5136, 1.7206]	1.207	[0.7303, 1.7342]	1.0039	[0.5773, 1.7379]	1.1606
	30	[0.7335, 1.3346]	0.6011	[0.7523, 1.7125]	0.9602	[0.8902, 1.6979]	0.8077	[0.7562, 1.7293]	0.9731
	50	[0.7402, 1.2963]	0.5561	[0.7563, 1.2112]	0.4549	[0.9103, 1.3302]	0.4799	[0.7623, 1.3194]	0.5571
PB	10	[0.7256, 1.6432]	0.9176	[0.6887, 1.8730]	1.1843	[0.7443, 1.9936]	1.2493	[0.6933, 1.8432]	1.1499
	20	[0.7312, 1.6324]	0.9012	[0.7809, 1.8566]	1.0757	[0.7892, 1.8952]	1.106	[0.8012, 1.8979]	1.0967
	30	[0.7418, 1.6330]	0.8912	[0.7942, 1.5653]	0.8711	[0.7902, 1.8146]	1.0244	[0.7995, 1.35669]	0.8674
	50	[0.7561, 1.5744]	0.8183	[0.8051, 1.26519]	0.8468	[0.7912, 1.5135]	0.7223	[0.8089, 1.5601]	0.7512
BCPB	10	[0.7220, 1.6042]	0.8822	[0.7523, 1.7325]	0.9802	[0.7902, 1.8979]	1.1077	[0.7562, 1.8283]	1.0721*
	20	[0.7524, 1.5761]	0.8237	[0.763, 1.8073]	1.0443	[0.8298, 1.8939]	1.0641	[0.7883, 1.7122]	0.9239*
	30	[0.7732, 1.5636]	0.7904	[0.7803, 1.6314]	0.8511	[0.8973, 1.8103]	0.913	[0.7903, 1.6832]	0.8929
	50	[0.7768, 1.5592]	0.7824	[0.7959, 1.5351]	0.7392	[0.9312, 1.7527]	0.8215	[0.7991, 1.6403]	0.8412*
PTB	10	[0.8851, 1.7562]	0.8711	[0.8023, 1.8962]	1.0939	[0.7776, 1.8792]	1.1016*	[0.7370, 1.8662]	1.1292
	20	[0.9213, 1.7175]	0.7962	[0.8532, 1.7308]	0.8776	0.8232, 1.7957]	0.9725*	[0.8109, 1.7892]	0.9783
	30	[0.9343, 1.708]	0.7737	[0.8832, 1.6998]	0.8166	[0.8454, 1.7379]	0.8925*	[0.8216, 1.6979]	0.8763
	50	[0.9580, 1.7078]	0.7498	[0.9052, 1.6881]	0.7829	[0.8523, 1.7159]	0.8636*	[0.8581, 1.6475]	0.7894

TABLE VI. ANOVA TABLE UNDER NORMAL DISTRIBUTION

Sources	Degree of freedom	SS	MS	F	P
n	3	1.68907	0.563023	10.62	0.003
Method	3	0.07372	0.024574	0.46	0.175
Error	9	0.47715	0.053017		
Total error	15	2.23994			
S = 0.2303 R-Sq = 78.70% R-Sq(adjust) = 64.50%					

TABLE VII. ANOVA TABLE UNDER *T5* DISTRIBUTION

Sources	Degree of freedom	SS	MS	F	P
n	3	0.85628	0.285428	17.93	0.000
Method	3	0.09789	0.032631	2.05	0.177
Error	9	0.14327	0.015918		
Total error	15	1.09744			
S = 0.1262 R-Sq = 86.95% R-Sq(adjust)= 78.24%					

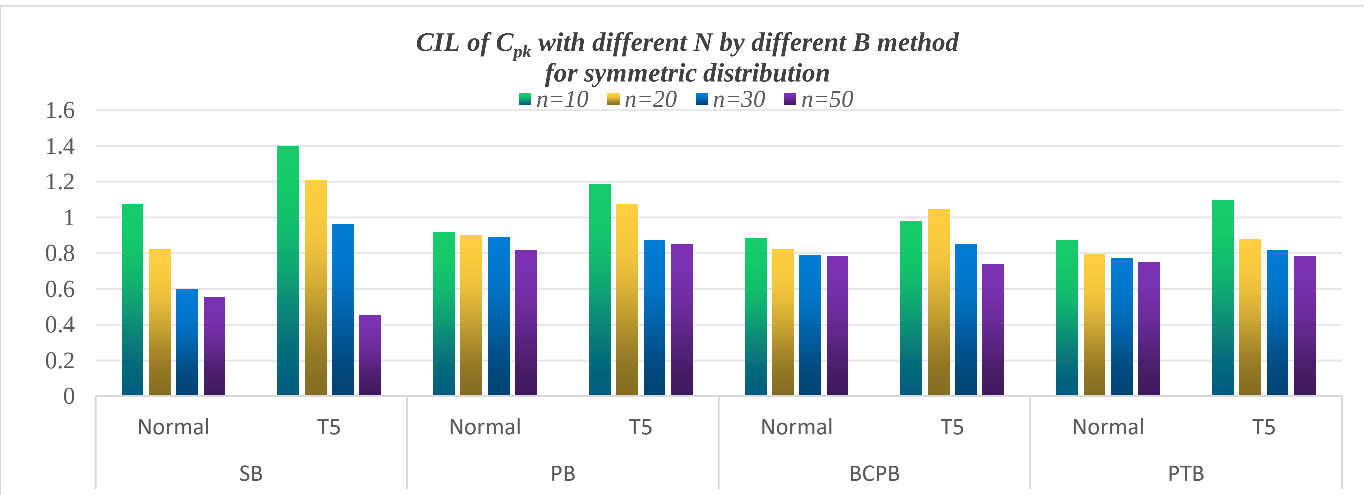

Fig. 1. CIL of *Cpk* under symmetrical distribution.

B. When the Data Follows Asymmetric Distribution

The ANOVA and image results are as follows: sample size n and Bootstrap methods are both significant factors affecting CIL (their P value are both less than 0.05). That is to say, the larger of sample size n is, the shorter of the CIL is.

Combined with Fig. 2, it can be found that the CIL under different Bootstrap methods are different but when the data follows χ_5^2 distribution, and the *CIL* of the later two Bootstrap methods is more stable, but the *CIL* of PTB is the shortest. While When the data follows *logn(0,0.4)* distribution (which is slightly skewed), the ANOVA and image results are a little different, the *CIL* of the BCPB methods is more stable and shorter. That is to say:

- In skew distribution, PTB method and BCPB method are better than SB and PB method in estimating *CIL*.
- In the slightly skewed distribution, such as *log-normal* (0, 0.4) distribution, BCPB method is recommended. For moderately skewed distributions, such as χ_5^2 distributions, the PTB method is recommended.

TABLE VIII. TWO FACTORS OF ANOVA TABLE UNDER CHI-SQUARE DISTRIBUTION

Sources	Degree of freedom	SS	MS	F	P
n	3	0.52345	0.174482	6.77	0.011
Method	3	0.32737	0.109124	4.23	0.040
Error	9	0.23212	0.025792		
Total error	15	1.08294			
S = 0.1606 R-Sq = 78.57% R-Sq(adjust) = 64.28%					

TABLE IX. TWO FACTORS ANOVA TABLE UNDER LOG-NORMAL*(0,0.4)* DISTRIBUTION

Sources	Degree of freedom	SS	MS	F	P
n	3	1.66676	0.555587	54.54	0.000
Method	3	0.15233	0.050778	4.98	0.026
Error	9	0.09168	0.010187		
Total error	15	1.91078			
S = 0.1009 R-Sq = 95.20% R-Sq(adjust)= 92.00%					

C. Findings

1) In symmetric distribution: Based on all the above ANOVA (see Tables VI-VII) and combined with Fig. 1, the following findings can be drawn:

Take normal distribution and *T* distribution as examples, only the sample size has a significant effect on the length of the confidence interval: the larger the sample size *n* is, the shorter the *CIL* is. In the symmetric distribution, the four Bootstrap methods had no significant effect on the length of confidence interval, that is, there was no significant difference between the four methods.

2) In asymmetric distribution: Based on all the above ANOVA (see Tables VIII-IX) and combined with Fig. 2, the following findings can be drawn:

Take chi-square distributions and log-normal distributions as examples, both the sample size *n* and the Bootstrap method are important factors affecting *CIL*. The *CIL* gets shorter as the sample size *n* increases. In addition, the confidence intervals calculated under the four Bootstrap methods are significantly different.

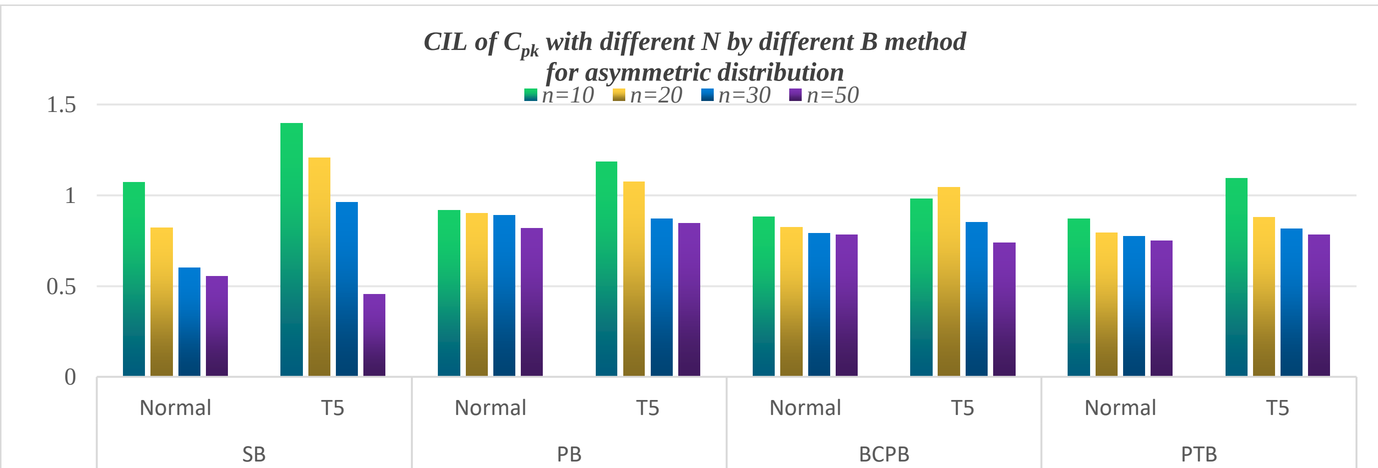

Fig. 2. CIL of Cpk under asymmetric distribution.

VI. Conclusion

This paper mainly studies the process capability index in the case of small samples. After analyzing the influence of sample size on the confidence interval of traditional process capability analysis index, we introduced Bootstrap method to solve and compare the confidence interval of process capability index *Cp* and *Cpk* in the case of small samples. ANOVA was used to verify the significant effects of sample size and different Bootstrap methods on confidence intervals. Some valuable findings were made:

1) In symmetric distributions, such as normal and *T*-distributions, only the sample size has a significant effect on the length of the confidence interval: the larger the sample size, the shorter the CIL. The Bootstrap method has no significant effect on the length of the confidence interval, that is, there is no significant difference between the four methods.

2) In asymmetric distributions, such as chi-square and lognormal distributions, both sample size and Bootstrap method are important factors affecting *CIL*. Combined with the variation of distribution skewness, the user can choose the appropriate Bootstrap method.

The above simulation and analysis only focus on the calculation and comparison of confidence intervals and their interval lengths, while the interval coverage ratio and standard difference of confidence intervals of PCI under these four methods have not been involved, and further in-depth research is needed.

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